EC 3210 Solutions

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Assignment 7

7.1 Consider the transcendental equation for G for a homogeneously broadened material (one line below Eq. 7.22). If $\beta_0 d = 1.386$, calculate the value of G for $I_{\text{in}}/I_s = 10^{-2}, 10^{-1}, 10^0, 10^1$ and 10^2.

We want to solve the following equation,

$$\ln(G) + \left(\frac{I_{\text{in}}}{I_s}\right)(G-1) = 1.386 \tag{1}$$

for $I_{\text{in}}/I_s = 10^{-2}$, 10^{-1}, 10^0, 10^1 and 10^2. The following table gives the results of using an equation solving routine for these values (and more).

I_{in}/I_s	G	G in dB
0.01	**3.89**	**5.89**
0.02	3.78	5.78
0.05	3.52	5.47
0.08	3.32	5.21
0.10	**3.21**	**5.06**
0.20	2.79	4.46
0.50	2.20	3.42
0.80	1.92	2.83
1	**1.80**	**2.55**
2	1.49	1.74
6	1.20	0.79
8	1.16	0.63
10	**1.13**	**0.52**
20	1.07	0.28
60	1.02	0.10
80	1.02	0.07
100	1.01	0.06

Figure 1 shows the plot of this data demonstrating the decreased gain as the input irradiance increases.

7.2 Using Eq. 7.26, find the transcendental equation that is to be solved for the gain, G, in inhomogeneously broadened materials.

The saturated gain coefficient is given by

$$\beta = \frac{\beta_0}{\sqrt{1 + \dfrac{I}{I_s}}}\,. \tag{2}$$

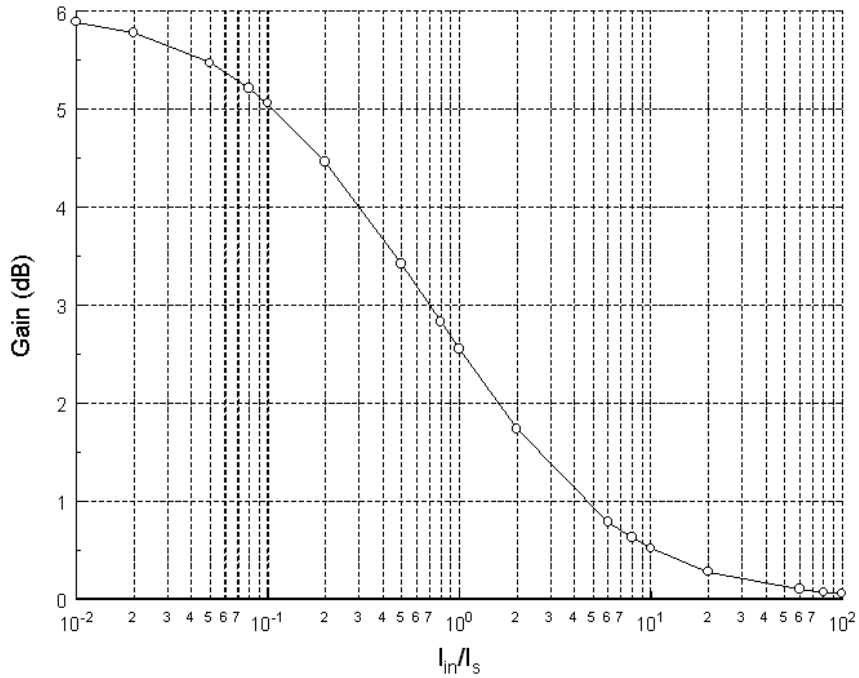

Figure 1: Saturated gain (in dB) vs. I_{in}/I_s for $\beta_0 d = 1.386$.

The expression for the incremental increase in irradiance, then, is

$$\frac{dI}{dz} = \beta I = \frac{\beta_0}{\sqrt{1 + \dfrac{I}{I_s}}} I = \frac{\beta_0 I \sqrt{I_s}}{\sqrt{I + I_s}} \tag{3a}$$

or

$$\frac{\sqrt{I + I_s}}{I\sqrt{I_s}}\, dI = \beta_0\, dz\,. \tag{3b}$$

Integrating both sides of the equation, we have

$$\int_{I=I(0)}^{I=I(d)} \frac{\sqrt{I + I_s}}{I\sqrt{I_s}}\, dI = \int_{z=0}^{z=d} \beta_0\, dz\,. \tag{4}$$

The right integral of Eq. 4 is $\beta_0 d$.

To evaluate the left integral, we find in Table of integrals (or from the Web integral engine offered by *Mathematica* at **www.integrals.com**) that

$$\int \frac{\sqrt{x + a}}{x}\, dx = 2\sqrt{x + a} - 2\sqrt{a}\tanh^{-1}\left(\sqrt{1 + \frac{x}{a}}\right)\,. \tag{5a}$$

We also have the identity that

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \tag{5b}$$

so,

$$\int \frac{\sqrt{x+a}}{x}\,dx = 2\sqrt{x+a} - \sqrt{a}\ln\left(\frac{1+\sqrt{1+\frac{x}{a}}}{1-\sqrt{1+\frac{x}{a}}}\right). \tag{5c}$$

We have $x = I$ and $a = I_s$, so

$$\frac{1}{\sqrt{I_s}}\int_{I=I(0)}^{I=I(d)} \frac{\sqrt{I+I_s}}{I}\,dI = \frac{2}{\sqrt{I_s}}\sqrt{I+I_s}\Big]_{I(0)}^{I(d)} - \frac{\sqrt{I_s}}{\sqrt{I_s}}\ln\left(\frac{1+\sqrt{1+\frac{I}{I_s}}}{1-\sqrt{1+\frac{I}{I_s}}}\right)\Bigg]_{I(0)}^{I(d)} \tag{6}$$

$$= 2\sqrt{1+\frac{I}{I_s}}\Bigg]_{I(0)}^{I(d)} - \ln\left(\frac{1+\sqrt{1+\frac{I}{I_s}}}{1-\sqrt{1+\frac{I}{I_s}}}\right)\Bigg]_{I(0)}^{I(d)} \tag{7}$$

$$= 2\sqrt{1+\frac{I(d)}{I_s}} - 2\sqrt{1+\frac{I(0)}{I_s}}$$
$$- \ln\left(\frac{1+\sqrt{1+\frac{I(d)}{I_s}}}{1-\sqrt{1+\frac{I(d)}{I_s}}}\right) + \ln\left(\frac{1+\sqrt{1+\frac{I(0)}{I_s}}}{1-\sqrt{1+\frac{I(0)}{I_s}}}\right).$$

We recognize that we want the final equation to be in terms of G ($G = I(d)/I(0)$) so, wherever we have $I(d)$, we replace it with $GI(0)$. Hence, Eq. 4 on the facing page is

$$2\sqrt{1+G\frac{I(0)}{I_s}} - 2\sqrt{1+\frac{I(0)}{I_s}} - \ln\left(\frac{1+\sqrt{1+G\frac{I(0)}{I_s}}}{1-\sqrt{1+G\frac{I(0)}{I_s}}}\right) + \ln\left(\frac{1+\sqrt{1+\frac{I(0)}{I_s}}}{1-\sqrt{1+\frac{I(0)}{I_s}}}\right) = \beta_0 d. \tag{8}$$

This equation can be solved for G vs. $I(0)/I_s$.