COMPUTER SIMULATION OF LINEAR ACOUSTIC DIFFRACTION

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ABSTRACT

Computer-aided acoustical imaging systems and computer simulations of other acoustic imaging techniques frequently require simulation of linear acoustic diffraction of large complex-valued data arrays. Computation efficiency requires the use of fast Fourier transform techniques. This paper compares two Fourier transform formulations of the propagation problem: the Fresnel integral and the spatial frequency domain approach. The following features are compared: restrictions on maximum and minimum propagation distances, sample sizes and number of samples required, adaptability to image processing techniques, and computational requirements.

INTRODUCTION

The use of computers in computer-aided acoustic imaging has become increasingly popular in recent years. The use of the computer in obtaining images by such techniques as backward wave propagation[1] offers such advantages as the elimination of the reconstruction wavelength scaling problem[2] obtained with optical reconstruction techniques, reference-free holography that uses the linear detection properties of piezoclectric transducers, and the possibility of incorporating image enhancement to improve the image obtained. Additionally computer simulation of the holographic process has been a useful tool in studying such novel techniques as phase-only holograms[3] and kinoforms[4]. In most computer-aided imaging techniques it is necessary to mathematically simulate the scalar wave diffraction process. This paper compares two techniques for this simulation, describing their features and some of their advantages and disadvantages.

In representing the acoustic diffraction formulations we will implicitly assume that the propagation medium is linear and homogeneous. We also will require that the resulting diffraction integrals must be amenable to computer solution in a reasonable amount of time which at the present infers that the integrals must be in the form of Fourier transforms so that the speed and efficiency of the Fast Fourier Transform[5] (FFT) can be brought to bear on the problem. The general problem then is: given a complex scalar wave $\underline{U}_i(x_i, y_i, 0)$ at some input plane, find an expression for the wave $\underline{U}_o(x_o, y_o; z)$ at some parallel output plane a distance z away, subject to the wave equations. Two forms of the solution incorporate the Fourier transform and will be considered after a short review of the features of the analog Fourier transform and the discrete Fourier transform (DFT).

The two dimensional analog Fourier transform is defined by the relationships

$$\underline{U}(x,y) = \iint_{-\infty}^{\infty} \underline{A}(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv = \mathcal{F}^{-1}\{\underline{A}(u,v)\} \tag{1}$$

$$\underline{A}(u,v) = \iint_{-\infty}^{\infty} \underline{U}(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy = \mathcal{F}\{\underline{U}(x,y)\} \tag{2}$$

where $\underline{U}(x,y)$ is the complex function in the space domain; $\underline{A}(u,v)$ is the complex transform in the spatial frequency domain; u,v are spatial frequencies (dimensions of cycles/meter); and $\mathcal{F}, \mathcal{F}^{-1}$ are symbolic operators for the transform and inverse transform operations respectively.

The discrete version of the Fourier transform (assuming an equal number of samples and sample spacing in both dimensions) is given by

$$\begin{aligned} \underline{f}(ma,na) &= \frac{1}{N^2}\sum_{k=o}^{N-1}\sum_{\ell=o}^{N-1} \underline{A}(k\Omega,\ell\Omega)\, e^{j2\pi\left[(ma\cdot k\Omega)+(na\cdot \ell\Omega)\right]} \\ &= F^{-1}\{\underline{A}\ (k\Omega,\ell\Omega)\} \end{aligned} \tag{3}$$

$$\begin{aligned} \underline{A}(k\Omega,\ell\Omega) &= \sum_{m=o}^{N-1}\sum_{n=o}^{N-1} f(ma,na)\, e^{-j2\pi\left[(ma\cdot k\Omega)+(na\cdot \ell\Omega)\right]} \\ &= F\ \{f(ma,na)\} \end{aligned} \tag{4}$$

where $o \leq m,n \leq N-1$;

$o \leq k,\ell \leq N-1$;

$\underline{f}$ (ma,na) is a complex valued sequence of samples in the space domain;
$A(k\Omega,\ell\Omega)$ is a complex valued sequence of samples in the spatial frequency domain;
N is the total number of samples in one dimension in the space or frequency domain (N x N total sample values);
a is the sample spacing in the space domain;
Ω is the sample spacing in the spatial frequency domain (and is equal to 1/Na);
F, F^{-1} are symbolic operators for the discrete Fourier transform and the inverse transform operation respectively.

The fast Fourier transform (FFT) is an efficient algorithm that uses symmetry properties to compute this discrete transform. The efficiency of this algorithm for large $N(\geq 8)$ makes it the only practical method of processing large two-dimensional complex valued data arrays, as is required in computer-aided acoustic imaging. Several properties of the DFT are mentioned here as they have important repercussions later.

1. As mentioned above, if the sample size in the space domain is a, then the sample size in the spatial frequency domain is 1/Na where N is total number of samples in one dimension.

2. There are N x N samples in the space domain covering a region Na x Na; there are also N x N samples in the frequency domain covering a region 1/a x 1/a.

3. The DFT assumes that the input sequence is periodic in both the x and y dimensions. Hence the input is considered an infinite two-dimensional periodic array (with period Na). Similarly the inverse DFT requires that the sequence in the spatial frequency domain also be periodic in both dimensions (with a period of 1/a).

4. The scaling factor of $1/N^2$ in the inverse transform should be noted to ensure computational accuracy.

5. The wave fields in diffraction patterns are usually centered on the propagation axis (as in Fig 1a) to take full advantage of symmetry. The usual DFT algorithm however usually works on a wave-field that lies in the first quadrant and produces the spectrum also in the first quadrant with the (0,0) frequency component at the origin (as in Fig. 1b). In order to apply the usual DFT to the centered wave without getting a linear phase shift in the transform domain that accompanies simple translation, a data shuffle [10] can be used. Based on the assumed periodicity of the input wave

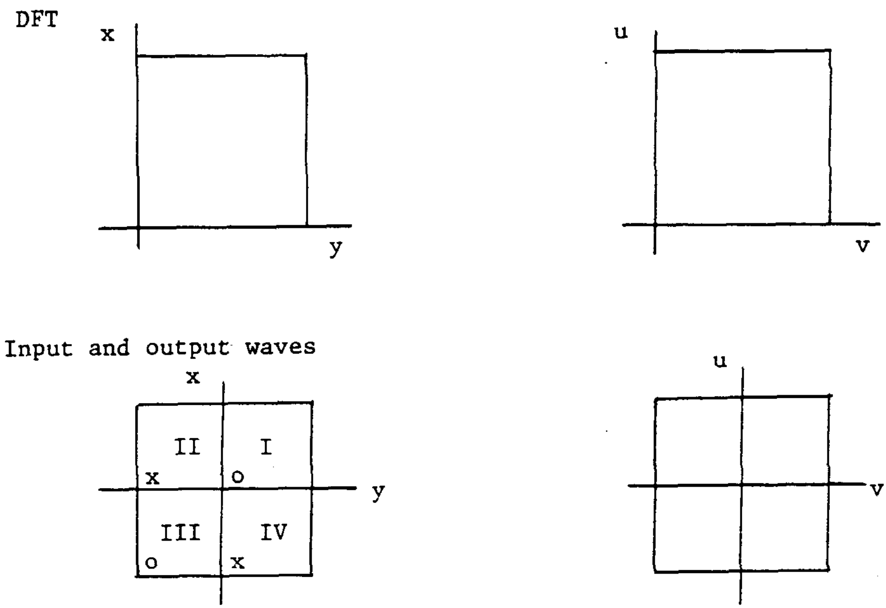

Figure 1 a) Geometric arrangement of input data and transform required by most discrete Fourier transform algorithms

b) Geometrical arrangement of input data and transform desired for diffraction problems. (Quadrant numbers and data location marking refer to data shuffle described in text.)

and the resulting symmetry, this shuffle applied both before and after taking any transform or inverse transform allows one to work with waves that are centered on the axis and FFT routines that work only in the first quadrant The shuffle routine exchanges the data in quadrants II and IV (see Fig 1b) and the data in quadrants I and III. (The alignment of the shuffle is indicated by the fact that the data of position x in quadrant II of Fig 1b is exchanged with the data of position x in quadrant IV. Similarly the data of the positions marked by the o's of quadrants I and III are also exchanged. Since this data shuffle is a mere exchange of data locations, the additional computation time is minimal except for the largest of arrays.

THE FRESNEL INTEGRAL

The first form of the solution of the propagation problem that incorporates the Fourier transform representation is the Fresnel integral[6]. Using the notation of the discrete Fourier transform this expression is:

$$\underline{U}_o(k\Delta x_o, \ell\Delta y_o) = \frac{e^{j\frac{2\pi z}{\lambda}}}{j\lambda z}\, e^{j2\pi\left[(k\Delta x_o)^2+(\ell\Delta y_o)^2\right]} \cdot F\left\{\underline{U}_i(ma,na)\, e^{j\frac{\pi}{\lambda^2}\left[(ma)^2+(na)^2\right]}\right\} \tag{5}$$

$$k\Omega = k\Delta x_o$$
$$\ell\Omega = \ell\Delta \chi_o$$

where $\underline{U}_i(ma,na)$ is the sampled input function (sample spacing $= a$);

z is the propagation distance;

Δxo, Δyo are sample spacings of the output wave and are each equal to $\lambda z/Na$;

$\underline{U}_o(k\Delta x_o, \ell\Delta y_o)$ is the sampled value of the output wave at the plane a distance z from the plane of the input wave.

This expression for $\underline{U}$ $(k\Delta xo, \ell\Delta y_o)$ is valid only for propagation distances such that

$$z^3 >> \pi N^4(\Delta x_o - a)^4 | 16\lambda \tag{6}$$

The important properties of this formulation of the diffraction integrals are noted below.

1. There is a limitation on the minimum propagation distance due to the inequality of Eq. 6. There is no limitation on the maximum propagation distance; in fact the Fresnel integral becomes the much easier to solve Fraunhofer diffraction integral[6] at very large propagation distances.

2. The sample spacing of the output becomes larger with increasing propagation distance since $\Delta x_o = \lambda z/Na$. This is helpful for diverging waves since the sample spacing spreads at the same rate as the diverging wave ensuring complete coverage of the diverging beam with a minimum number of sample points.

3. The diffraction operation requires N^2 complex multiplications, one data shuffle, one FFT operation, another data shuffle, and another N^2 complex multiplications.

4. The required sample spacing is determined by frequency aliasing considerations[7]. Since the exact derivation of the number of samples depends on the object, we choose a representative one-dimensional object, a slit of width 2a, and find an estimate of the sample number as a rough guideline. The object is a slit of width 2a in a region 2W wide (see Fig 2a); the transform of this object is 2a sinc 2au (see Fig 2b). The spacing in the frequency domain will be $\Delta u = 1/2W$ and, if there are N samples, the maximum frequency will be

$$U_{max} = \frac{N}{2}\Delta u = \frac{N}{4W} \tag{7}$$

The energy contained in the frequencies above Umax will be aliased back into the frequencies below Umax. The fraction, ε, of the total energy that is aliased is

$$\varepsilon = \frac{1}{a}\int_{U_{max}}^{\infty} |A(u)|^2 du \simeq \frac{2W}{\pi^2 Na} \tag{8}$$

and hence the sample width is

$$\Delta x = \frac{2W}{N} = \pi^2 a\varepsilon \tag{9}$$

and the number of samples is

$$N = \frac{2W}{\pi^2 a\varepsilon} \tag{10}$$

Typically $\varepsilon \sim 5\%$ or 10% is used. Hence the amount of aliasing tol-

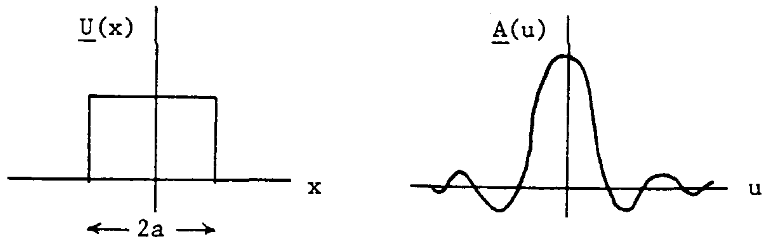

Figure 2 a) Hypothetical one dimensional object (a slit)
b) Transform of object

erated determines the sample spacing and the number of samples. As indicated previously this result is based on a particular input object (i.e. a slit). Objects with smaller scale discontinuities or perturbations might require finer sample spacing while objects with more gradual discontinuities can be analyzed with coarser sampling. However the results of Eqs. 9 and 10 give approximate values for sample spacing and total number of samples.

SPATIAL FREQUENCY DOMAIN APPROACH

The spatial frequency domain solution[6] to the diffraction problem relates the transform of the output wave to the transform of the input wave by a simple complex multiplication:

$$\underline{A}_o(k\Omega,\ell\Omega;z) = \underline{A}_i(k\Omega,\ell\Omega;0)\,e^{j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda k\Omega)^2-(\lambda k\Omega)^2}} \tag{11}$$

Realizing that the square root expression is imaginary for $(k\Omega)^2 + (\ell\Omega)^2 \geq {}^1/\lambda^2$ and that the contribution from these terms (the "evanescent waves") will be negligible if the propagation distance is more than several wavelengths we can simplify this expression to

$$\underline{A}_o\cdot(k\Omega,\ell\Omega;z) = \begin{cases} \underline{A}_i(k\Omega,\ell\Omega;o)\,e^{j\frac{2\pi z}{\lambda}\sqrt{1-(\lambda k\Omega)^2-(\lambda\ell\Omega)^2}} & \text{when } (k\Omega)^2+(\ell\Omega)^2 \leq \frac{1}{\lambda^2} \\ 0 & \text{when } (k\Omega)^2+(\ell\Omega)^2 \geq \frac{1}{\lambda^2} \end{cases} \tag{12}$$

where

$$\underline{A}_i(k\Omega, \ell\Omega; o) = F\{\underline{U}_i(ma,na)\};$$

$$\underline{A}_o(k\Omega, \ell\Omega; z) = F\{U_o(ma,na)\};$$

a is the spatial sample spacing;
Ω is the frequency domain sample spacing (equals 1/Na).

The properties of this frequency domain appraoch are as follows:

1. The sample spacing in the output plane is the same as that of the input wave x ($\Delta x_o = \Delta x_i = a$). For a diverging wave we would need many more samples in the output plane to adequately describe the wave because of its larger size, so we must either have numerous zero valued samples at the input plane to adequately cover the output wave or we must restrict our coverage of the output wave to only a small portion of its breadth. Fortunately a remedy has been found to this dilemma by Sziklas and Siegman.[7] Reference 7 presents a wave transformation that converts a diverging wave diffraction problem into a collimated wave problem. The collimated wave problem is adequately handled in the frequency domain approach by the coordinate systems having equal spacings in the input and output planes. The solution to the collimated beam problem may then be used to easily find the solution to the diverging beam problem. The net effect is to obtain an effective output plane sample spacing that expands with propagation distance so that a conservative number of sample points can adequately describe both the input wave and the output wave.

2. Because the DFT assumes that the input wave samples are repeated in a periodic two-dimensional array, at some propagation distance L, the waves from the other "objects" will overlap the wave from the original object, thereby limiting the maximum propagation distance for which the frequency domain approach can be used. Figure 3 and the following analysis uses the one-dimensional slit as an example of this effect and an estimate of the maximum propagation distance. Using an approximation to the Fresnel integral for this specific object, Sziklas and Siegman[7] show that allowing ε_1% of the total wave energy in the overlapping fields at a propagation distance L requires a guard band or region of zero valued samples (as in Fig. 3) of value

$$G \geq 1 + \frac{L\lambda}{2\pi^2 a\varepsilon_1} \tag{13}$$

This equation can also be used to find the maximum propagation distance L, given an object with a certain guard band value G and an allowed amount of energy overlap (e.g. 5%). Again this result is based on the slit object and would have to be increased for objects

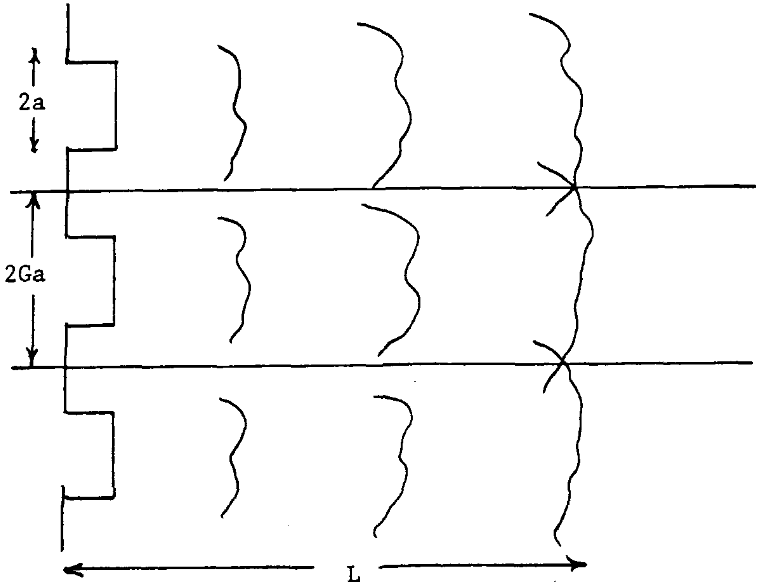

Figure 3 Hypothetical one dimensional slit object and its nearest periodic neighbors. At distance L the diffracted wavefronts are significantly overlapped.

with smaller features that would have to be resolved or decreased for objects with more tapered features than a slit.

3. It is noted from Eq. 12 that the output frequency spectrum is band-limited i.e. all frequency samples lying outside of a circle in the frequency domain with a radius of $1/\lambda$ are equal to zero (for propagation distances longer than several wavelengths). Hence only those samples of the input wave spectrum that lie within this same circle must be taken. This leads to a determination of the optimum spatial sample spacing and the fact that there is no frequency aliasing in the spatial frequency approach to the diffraction problem. Rather than restricting our frequency samples to those lying within the circle of radius $1/\lambda$ it is geometrically simpler to consider those lying in the rectangle $|u|<1/\lambda$ and $|v|<1/\lambda$ (as in Fig 4). This leads to an oversampling by 12% of the absolute minimum number of samples.

Choosing this sampling limit gives a maximum frequency in the

$$N = \frac{2Ga}{\Delta x} = \frac{4\,Ga}{\lambda} \tag{18}$$

It should be noted that many FFT routines require that the number of samples be an integral power of 2. Hence the final value of N may be that power of two above or below the calculated value of Eq. 18. If larger the field will be oversampled (with a resulting loss of computing efficiency and longer running times) or undersampled (with shorter running time but a loss of some resolution).

5. Another property of working in the frequency domain to handle the diffraction problem is that this approach is easily amenable to frequency domain image processing techniques[8] such as Weiner filtering, edge enhancement, deconvolution of the point spread function of the receiver, etc.). Here the spectrum of the diffracted wave can be manipulated by multiplication with a filter function:

$$\underline{A}_o'(u,v) = \underline{A}i(u,v)\ \underline{H}\ \text{prop}(u,v)\ \underline{H}\ \text{filter}(u,v) \tag{19}$$

where.

$\underline{A}_o'(u,v)$ is the spectrum of the processed output wave;

$\underline{H}$ prop(u,v) = $A_o(u,v)|Ai(u,v)$ is the "transfer function" for linear scalar diffraction and is found by dividing Eq. 12 by Ai(u,v);

$\underline{H}$ filter(u,v) is the filter function to perform the desired operation.[9]

Since one of the primary advantages of computer-aided imaging is the flexibility and capability to enhance the image and extract information, the ease of incorporating this class of frequency domain operations is a major advantage of the frequency domain approach.

SUMMARY

Table I summarizes the relative advantages and disadvantages of the two diffraction approaches. With an awareness of these strengths and weaknesses the researcher attempting computer-aided imaging will be able to choose the technique most suitable for his application.

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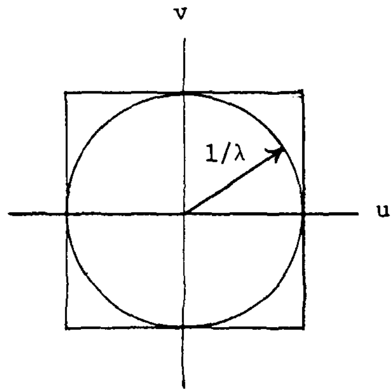

Figure 4 Frequency domain representation of bandlimited propagation showing circular limit (radius equals $1/\lambda$). Circumscribed rectangle shows geometrically simpler bandlimiting (with 12% oversampling).

U direction of

$$u_{max} = \frac{N}{2} \Delta U = \frac{1}{\lambda} \tag{14}$$

where Δu is the spatial frequency sample spacing. Hence,

$$\Delta u = \frac{2}{N\lambda} \tag{15}$$

and is also given by

$$\Delta u = \frac{1}{N\Delta x} \tag{16}$$

where Δx is the spatial sample spacing. Therefore

$$\Delta x = \frac{\lambda}{2} \tag{17}$$

is the optimum sample spacing of the object. It is noted that this sample spacing is that required to give resolution of $\lambda/2$, the "diffraction limited" resolution.

4. Knowing the required quard band size and the optimum sample spacing from properties 2 and 3 above it is now possible to compute the total number of samples by dividing the total object width (including the guard band) by the optimum spacing:

TABLE I Summary of comparison between diffraction techniques

FRESNEL INTEGRAL	FREQUENCY DOMAIN APPROACH
1. Expanding sample spacing for diverging wave.	1. Expanding sample spacing can be made to occur by wave transformation.
2. Limited minimum diffraction distance.	2. Unlimited minimum diffraction distance.
3. Unlimited maximum diffraction distance.	3. Requires large guard band (and more samples) for longer diffraction distances.
4. Does not predict diffraction limited diffraction.	4. Correctly predicts diffraction limited resolution.
5. Some frequency aliasing.	5. No frequency aliasing due to bandlimited nature.
5. Image processing is separate operation.	6. Frequency domain filtering techniques are easily incorporated.
7. Requires two N^2 complex multiplications, two data shuffles, and one FFT operation.	7. Requires N^2 multiplications, four data shuffles, and two FFT operations.

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